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Fano threefolds with affine canonical extension

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Conjecture (Campana–Peternell)

M a Fano manifold. Then T_M is nef iff M is rational homogeneous manifold.

Remark:
- Known if $\dim M \le 5$ (Kanemitsu, Hwang–Mok, ...)
- Partial results in higher dimension (Occhetta, Solá Conde, Watanabe, Wiśniewski)

Today: alternative point of view on positivity of T_M.

$\rightsquigarrow$ Object: canonical extension of T_M.

$\alpha \in H^1(M, \Omega_M) \simeq \mathrm{Ext}^1(T_M, \mathcal{O}_M)$

ample class

$\rightsquigarrow$ extension associated to α: $0 \to \mathcal{O}_M \to V \to T_M \to 0$

$\Rightarrow$ inclusion $\mathbb{P}(T_M) \subset \mathbb{P}(V)$

Note that $N_{\mathbb{P}(T_M)/\mathbb{P}(V)} \simeq \mathcal{O}_{\mathbb{P}(T_M)}(1)$ ← tautological bundle on $\mathbb{P}(T_M)$

Technical approach: try to relate positivity of $N_{\mathbb{P}(T_M)/\mathbb{P}(V)}$ with the geometry of $Z_M = \mathbb{P}(V) \setminus \mathbb{P}(T_M)$

We call Z_M the canonical extension of M (Greb–Wong 2020)

Observation (Greb–Wong)

Z_M does not contain any projective subvarieties (of pos. dimension)

Conjecture (H, Peternell)

M a Fano manifold. Then

Z_M is affine iff T_M is nef & big.

Remark: „⇐" is obvious. Proof:

T_M is nef & big $\overset{\text{defn}}{\Leftrightarrow}$ $\mathcal{O}_{\mathbb{P}(T_M)}(1)$ is nef and big $\Leftrightarrow$ $\mathcal{O}_{\mathbb{P}(V)}(1)$ is nef and big

$\mathbb{P}(V) \to M$

$-K_{\mathbb{P}(V)} = (\dim M + 1)\, \mathcal{O}_{\mathbb{P}(V)}(1)$ ~~is a weak~~

$\Rightarrow \mathbb{P}(V)$ is a weak Fano manifold.

$\rightsquigarrow$ Have a birational morphism $\mu: \mathbb{P}(V) \to \mathbb{P}(V)_{\text{anti}}$ (anticanonical model).

Computation: μ maps $\mathbb{P}(T_M)$ onto a hyperplane section H.

and $Z_M \simeq \mathbb{P}(V)_{\text{antican}} \setminus H$

projective − hyperplane = affine. ∎

Main theorem (H, Peternell 2022)

The conjecture is true for Fano threefolds.

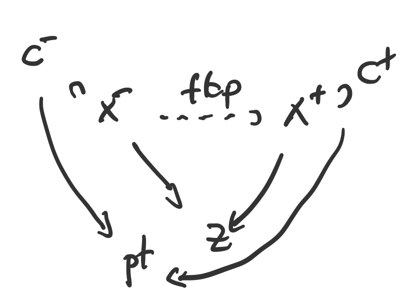
Initial evidence: Goodman's theorem

X projective manifold and $Y \subset X$ a smooth connected divisor s.t. $X \setminus Y$ is affine. Then $N_{Y/X}$ is big.

Unfortunately it is not true that $X \setminus Y$ affine implies $N_{Y/X}$ is nef.

Example (H, Peternell 2021, Ottem 2012)

X^- smooth projective threefold
$\cup$
Y^- smooth ample divisor
$\cup$
C^- curve that can be flopped.



Let $Y^+ \subset X^+$ be the strict transform of $Y^- \subset X^-$. Then

$X^+ \setminus Y^+ \simeq X^- \setminus Y^-$ affine

but Y^+ is not nef because $Y^+ \cdot C^+ < 0$.

Idea: use that $X = \mathbb{P}(V)$ and $Y = \mathbb{P}(T_M)$.

Lemma: M projective manifold s.t. Z_M is affine.

Then M is not a blowup $Bl_p M'$ with M' a projective manifold.

~~Proof~~ Q: how do we use Z_M affine?

A: affine manifolds have many holomorphic functions!

More precisely: given any discrete sequence of points $(z_n)_{n \in \mathbb{N}}$

$\exists\, f: Z_M \to \mathbb{C}$ holom. fctn s.t. $|f(z_n)| \xrightarrow[n \to \infty]{} \infty$

Proof of the lemma: $\mu: M \to M'$ blowup, $E \subset M$, $E \to p$, $p \in M'$

extension $0 \to \mathcal{O}_M \to V \to T_M \to 0$

induces extension $0 \to \mathcal{O}_{M'} \to V' \to T_{M'} \to 0$

not well defined over E.

$$\begin{array}{ccc} Z_M & \dashrightarrow & Z_{M'} \\ \pi \downarrow & & \downarrow \pi' \\ M & \longrightarrow & M' \end{array}$$

but $Z_M \setminus \pi^{-1}(E) \simeq Z_{M'} \setminus \pi'^{-1}(p)$

(affine ↓ Z_M; $Z_M \setminus \pi^{-1}(E)$: still have many holom. fctns.)

Choose $f: Z_M \setminus \pi^{-1}(E) \to \mathbb{C}$ holom. that goes to ∞ near $\pi^{-1}(E)$.

$\cong Z_{M'} \setminus \pi'^{-1}(p)$ ($\pi'^{-1}(p)$ has codim ≥ 2 in $Z_{M'}$)

$\overset{\text{Hartogs}}{\Rightarrow}$ f extends to $\bar{f}: Z_{M'} \to \mathbb{C}$

there $\bar{f}$ can't go to ∞ near the boundary ↯ ∎

Further evidence: T_M big itself is a very restrictive property.

Corollary of thm of Höring–Jie Liu:

M smooth Fano threefold with $\rho(M) = 1$. Then

T_M is big iff $M \simeq$
- $\mathbb{P}^3$
- Q^3 quadric

} rational homogeneous.

- V_5 del Pezzo threefold of degree 5.

(V_5 ↑ not rational homogeneous, T_M not nef.)

Question: ~~Idea:~~ Is V_5 a counterexample to our conjecture?

Idea: construct a weak Fano model of $\mathbb{P}(T_{V_5})$.

ψ: $X := \mathbb{P}(V) \overset{\text{sequence of antiflips}}{\dashrightarrow} X^-$ terminal, $\mathbb{Q}$-factorial sing.

$\cup$ $\qquad\qquad \cup$

$Y := \mathbb{P}(T_M) \overset{\text{sequence of antiflips}}{\dashrightarrow} Y^-$ s.t. X' is weak Fano

$\downarrow$ $\qquad$ i.e. $-K_{X'}$ is nef & big

$M = V_5$ $\qquad$ and $\psi|_{Z_M}$ is an embedding.

$-K_X$ is big.

As in first proof it is obvious that $X^- \setminus Y^-$ is affine.

and $\psi: Z_M \hookrightarrow X^- \setminus Y^-$ (affine)

Big surprise: $\psi(Z_M) \subsetneq X^- \setminus Y^-$ and the complement has codim $= 2$

$\overset{\text{Hartogs}}{\Rightarrow}$ Z_M is not affine.

Proof of main theorem: $\rho(M) > 1$ restrictions on birat. geom + classification

$\rho(M) = 1$ by H–Liu we only have to check V_5.

For V_5 we construct the weak Fano model by hand

It's tedious, use projective geometry of V_5 (based on beautiful book by Cheltsov & Shramov) ∎

Final question: M a Fano s.t. T_M is big.

Can we prove existence of a weak Fano model for $\mathbb{P}(T_M) \subset \mathbb{P}(V)$?